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QUANTITATIVE EFFECT OF A FLEXIBLE FUSELAGE OF THE SYMMETRIC TORSIONAL MODES OF THE WINGS OF A LARGE AIRPLANE

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SUMMARY

Using Holzer's method of frequency calculation, the natural frequencies for the first two modes of torsional vibration of the wing were determined for a representative conventional airplane (B24-C) in the customary manner, the fuselage being considered as a rigid body. Next, using a method developed by N. O. Myklestad of the Guggenheim Aeronautics Laboratory at the California Institute of Technology, combined with Holzer's method, the natural frequencies for the same two modes of vibration were again determined, but with the fuselage this time being considered as flexible.

A comparison of results of the two methods indicates that in considering the fuselage as being flexible, a decrease in the natural frequency of torsional vibration may be expected. For the particular airplane selected, this decrease amounted to 6.68% for the first mode of vibration and to 39.1% for the second.

The investigation reported in this paper was entirely theoretical and was performed during the 1943-1944 school year at the Guggenheim Aeronautics Laboratory at the California Institute of Technology, Pasadena, California under the direction and supervision of Dr. N. O. Myklestad, research associate in aeronautics at the Institute.

INTRODUCTION

In the design of modern aircraft for higher and higher speeds, the designers are becoming increasingly more interested in the problems of flutter and vibration. One of these problems is that of the torsional vibration of the wings, which is dependent upon a number of factors, such as (1) the mass distribution both spanwise and chordwise of the wings themselves and of all units supported either on them or within them, (2) the torsional stiffness of the wings, (3) the torsional moment applied to the wings at the shifting center of pressure by the air loads, (4) the coupling between the wings in bending and the wings in torsion, (5) the torsional moment applied at the root of the wings by a flexing fuselage, and (6) the effect of compressibility as local velocities over the wing approach the velocity of sound. It is believed to be common practice in the aircraft industry generally to consider all but the last two of the factors enumerated above.

This paper then has as its objective the quantitative determination of the effect on a representative large airplane of the fifth factor enumerated above, namely, the effect of the torsional moment applied at the root of the wings by a flexing fuselage on the natural frequencies of the wing torsional vibration.

DEFINITION OF SYMBOLS

m_n - Fuselage mass concentrated at any station n, lbs. seconds squared per inch.

n - Number of any wing or fuselage station.

I - Mass moment of inertia of wing in inch lbs. seconds squared or bending moment of inertia of fuselage in inches to the fourth power.

I_o - Convenient reference value of bending moment of inertia for fuselage as a whole.

E - Modulus of elasticity of fuselage bending material in lbs. per square inch.

β_n - Angular deflection of wing at any station n in radians.

ω - Frequency of vibration in radians per second.

ℓ_n - Panel length of wing or fuselage between stations n and $n+1$ in inches.

S_n - Shear at fuselage station n in lbs.

M_n - Bending moment at fuselage station n in inch lbs.

M_b - " " of fuselage tail at elastic axis.

M_b' - " " " " nose " " ".

α_n - Slope of fuselage axis at any station n.

α_b - " " " tail at elastic axis.

α_b' - " " " nose " " ".

y_n - Deflection of fuselage at any station n in inches.

y_b - " " " tail at elastic axis.

y_0' - Deflection of fuselage nose at elastic axis.

v_{F_n} - Change in slope from n to $n+1$ due to a unit force at n.

v_{M_n} - " " " " n " $n+1$ " " " " moment at n.

d_{F_n} - " " deflection from n to $n+1$ due to a unit force at n.

d_{M_n} - " " " " n " $n+1$ " " " " moment at n.

ϕ - Slope of fuselage axis at extreme end of tail or nose.

h_n , f_n - Coefficients appearing in equation for α_n.

g_n , k_n - " " " " " y_n.

M_c - Total coupling moment introduced into wing by fuselage at the elastic axis.

$a_n = I_n / I_0$ - Non-dimensional symbol for fuselage bending moment of inertia at any station n.

$b_n = a_{n+1} - a_n = \dfrac{I_{n+1} - I_n}{I_0}$ - Non-dimensional symbol for increase in fuselage bending moment of inertia from station n to station $n+1$.

ξ - Variable distance from station n to any point in panel ℓ_n (between n and $n+1$).

$$K_n = m_n \omega^2 k_n \qquad K_n' = \sum_{i=1}^{i=n-1} \left[\ell_i \sum_{s=1}^{s=i} K_s \right]$$

$$G_n = m_n \omega^2 g_n \qquad G_n' = \sum_{i=1}^{i=n-1} \left[\ell_i \sum_{s=1}^{s=i} G_s \right]$$

k_n' - Torsional rigidity of wing in lb. inches per radian.

ANALYSIS

Because of the nature of this investigation and the difficulties involved in the measurement of the effect of one factor at a time on the torsional vibration of a wing, no experimental work was undertaken. Instead, the authors approached the problem from a purely theoretical viewpoint, and the investigation was performed entirely on that basis.

After a representative airplane for the investigation had been selected, it was necessary first to obtain the following information concerning it:

A. The wing (Table I), considering its mass and the masses of all bodies either attached to it or stored within it as being concentrated at a number of stations along its span:

   (1) the distance of each station from the wing root in inches,

   (2) the mass polar moment of inertia I_n about the elastic axis of the wing of the mass considered to be concentrated at each station in lb-inches seconds squared, and

   (3) the rigidity k_n' in lb-inches per radian, or its reciprocal, of the wing in torsion between each station.

B. The fuselage (Table II), considering its mass and the masses of all bodies either attached to it or stored within it as being concentrated at a number of stations along its length:

(1) the distance of each station from the fuselage nose in inches,

(2) the bending moment of inertia about a horizontal axis perpendicular to the longitudinal axis of the fuselage I_n in inches to the fourth power, and

(3) the total mass m_n considered as concentrated at each station in lbs. seconds squared per inch.

After receipt of the required information for the wing, it was possible to calculate the natural frequencies of the wing in torsion for as many modes of vibration as were desired, considering the wings as being built in to a stiff fuselage with an extremely high moment of inertia compared with that of each station along the wing. This calculation was actually carried out for two modes of vibration following Holzer's method as outlined on pages 228 and 229 of Ref. 1, an example of which has been appended to this paper as Table III with an explanation included in the appendix. The results of this calculation have been tabulated in Table IV and plotted on Fig. 1, and show that the natural frequencies for the first two modes as determined by this calculation are 33.67 and 71.70 radians per second respectively.

This completed the first phase of the investigation; and with the required information for the fuselage then at hand, it was possible to proceed with the second phase, namely, the calculation of the natural frequencies of the wing in torsion for as many modes of vibration as were desired, considering the torsional moment applied at the root of the wings by a flexing fuselage. The problems immediately con-

fronting the authors in this phase of the investigation were those of determining (1) the torsional moment produced at the root of a wing by a flexing fuselage and (2) the method of coupling this moment into the wing at its root.

For the solution of the first of these problems a method developed in Ref. 2 for the antisymmetric bending of wings was applied to the flexing fuselage, considering the fuselage to be made up of two independent beams extending in opposite directions from the location of the elastic axis at the root of the wing. This method has the advantage of yielding immediately the bending moment at any particular station along a cantilever beam and the slope of the beam at that station as linear functions of the normal displacement of the beam. Consequently, the procedure followed was, first, to calculate the bending moments and the slopes, at the location of the elastic axis at the root of the wing, of both the portion of the fuselage aft of this location and the portion of the fuselage forward of this location resulting from a unit downward displacement of the extreme end of both the tail and the nose. The bending moment at the elastic axis and the slope at that location of the after portion of the fuselage were designated as M_b and α_b respectively, and of the forward portion of the fuselage as M_b' and α_b' respectively.

Next, since the fuselage is actually a continuous structure throughout its length, its slope on either side of the elastic axis must equal the slope on the other side of the elastic axis. This leads to the result that, since the initial displacements of both the

tail and the nose were taken to be positive downwards, in order for the slope forward of the elastic axis to equal that aft of the elastic axis, the slope forward of the elastic axis α_b' must be multiplied by the ratio $(-\alpha_b/\alpha_b')$. This same result would have been obtained had the initial displacement of the nose been multiplied by this ratio $(-\alpha_b/\alpha_b')$; and since the bending moment developed is a linear function of the displacement of the free end, the bending moment produced at the elastic axis by the forward portion of the fuselage M_b' should also be multiplied by this same ratio. We then have that, for the continuous fuselage, the bending moments at the elastic axis due to the after and forward portions of the fuselage are given by the expressions, M_b and $(-\alpha_b/\alpha_b') \times M_b'$ respectively. However, these two components oppose one another; consequently, in order to determine the total bending moment M_c from the fuselage to be coupled into the root of the wing, one must be subtracted from the other. If the direction of M_b is taken to be the positive direction, it may readily be seen then that $M_c = M_b - (-\alpha_b/\alpha_b')M_b' = M_b + (\alpha_b/\alpha_b')M_b'$. And since the bending moment from the fuselage at the elastic axis enters the wing as a torsional moment, M_c is the torsional moment produced at the root of the wing, the amount of it entering each side of the wing being $\frac{1}{2}M_c$, assuming symmetrical twisting of the wing.

For the method of coupling this moment into the wing at its root, one side of the wing was considered as a free body in torsion with this torsional moment of $\frac{1}{2}M_c$ applied at its root. Assuming arbitrary unit angular deflections of the wing tip, the Holzer's calculations made

during the first phase of this investigation yielded the following information for each frequency selected:

(1) The torsional moment developed within the wing at the first station outboard of the root due to the rotational inertia forces within the wing ($\sum_{n=1}^{n=6} I_n \omega^2 \beta_n$ from Table III), and

(2) The angle of twist developed at the root of the wing (β_1 from Table III).

Since in this calculation the torsional moment developed at any particular station and the angle of twist at that station are given as linear functions of the arbitrary angular deflection of the wing tip, any desired angle of twist at the wing root can be obtained by properly adjusting the arbitrary angular deflection of the wing tip. Since the wing can be considered to be built-in to the fuselage, its angle of twist at the root should equal the slope of the fuselage at the wing's elastic axis, and in order to obtain this angle of twist at the root it is necessary to multiply the original arbitrary angular deflection of the wing tip by the ratio (α_b/β_1). Having multiplied the original arbitrary angular deflection of the wing tip by this ratio, it is then necessary to multiply the torsional moment developed within the wing at the first station outboard of the wing root by this ratio also. Hence, this moment is then found to equal $(\alpha_b/\beta_1)\sum_{n=1}^{n=6} I_n \omega^2 \beta_n$, and adding this to the torsional moment applied at the root of the wing by the fuselage, a residual torsional moment or torque on the wing of $M = \frac{1}{2}M_c + (\alpha_b/\beta_1)\sum_{n=1}^{n=6} I_n \omega^2 \beta_n$ is found. This residual torque is then the additional applied moment required in order to force

the wing to vibrate at the assumed frequency. This value of the residual torque is then plotted against the assumed frequency; and the process is then repeated for other assumed frequencies, one point on the plot being obtained for each assumed frequency. A complete example of the calculation by this method for one assumed frequency has been appended to this paper as Tables V(a), V(b), VI(a), VI(b) and VII, with a brief explanation of them included in the appendix.

The results of this calculation have been tabulated in Table VIII.

After a sufficient number of points have been obtained, a curve may be drawn through them as has been done in Fig. 1. Again the points at which this curve crosses the frequency axis determine the natural frequencies of torsional vibration for the wing, for at these points the residual torque becomes zero, and hence the additional applied moment required to force the wing to vibrate at that frequency also becomes zero. From Fig. 1 it can be seen that the natural frequencies for the first two modes as determined by this calculation are 31.42 and 43.65 radians per second respectively. (See Table IX for tabulation of final results.). These are reductions of 6.68% and 39.1% respectively from the frequencies of the first two modes found in the first phase of this investigation. Accordingly, it may be concluded that, whereas the consideration of a flexing fuselage has a small but appreciable effect on the frequency of the first mode of torsional vibration of the wing, it has a very decided effect in lowering the frequency of the second mode. Probably, considering the trend of the curves of Fig. 1, this same effect is carried on in pro-

gression to subsequent modes of vibration; hence it is the studied opinion of the authors that this effect should be considered in the calculation of the natural torsional frequencies of the wing.

The closing phase of this investigation was the determination of the fuselage deflection curves for each of the two modes of vibration determined above. This was accomplished with facility from the calculations involved in the determination of the bending moments and slopes of the forward and the after portions of the fuselage in the second phase of this investigation. The deflection at any station of the fuselage is designated as y_n, and columns so headed may be found in both Tables VI (a) and VI(b). Again, the values of y_n' given in Table VI(a) must be multiplied by the ratio $(-\alpha_b/\alpha_b')$ in order to give them the correct magnitude with respect to those given in Table VI(b). Fuselage deflections are tabulated in Table X. A plot of the values of y_n calculated for the two natural frequencies found in the second phase of this investigation was made and has been appended to this report as Fig. 2. A perusal of this figure will indicate that the deflection curves for the fuselage for the two modes of wing torsional vibration are very similar, there being no reflex curvatures along the fuselage length in either case. In the first mode the nose deflection is about one-seventh that of the tail whereas in the second mode it is almost twice that of the tail, from which it can be seen quite readily that for a given deflection of the nose the fuselage curvature will be much greater for the first mode than it will for the second.

The relative angular deflections of the wing at each station along its span can be determined very readily by referring to the columns headed β in Table III for the calculation for a rigid fuselage and in Table VII for the calculation for a flexing fuselage. These values must be multiplied by the ratio (α_b/β_T) for each frequency selected in the calculation for a flexing fuselage, as has already been done in the determination of the residual torque acting on the wing, in order to determine the actual magnitudes corresponding to a unit downward deflection of the tail. This has been done and the results for the two modes tabulated in Table XI and plotted in Fig. 3.

Fig. 6 is a schematic illustration of the two modes of vibration, assuming a unit downward deflection of the extreme tail in each case.

CONCLUSIONS

In the case of the airplane investigated herein, the consideration of a flexing fuselage has a small but appreciable effect on the frequency of the first mode of torsional vibration of the wing, but it has a very decided effect in lowering the frequency of the second mode.

The deflection curves for the fuselage for the first two modes of wing torsional vibration are very similar, there being no reflex curvatures along the fuselage length in either case. However, for a given deflection of the nose, the fuselage curvature will be much greater for the first mode than it will be for the second.

It must be understood that the above conclusions apply only to the particular airplane which has been investigated herein. This paper is not submitted with the intent to show that effects of similar magnitude can be expected for all airplanes, but simply that the effect should be investigated with the thought in mind that it might prove appreciable, particularly in the case of higher modes.

APPENDIX

I. CALCULATION FOR THE RIGID FUSELAGE

The method used is outlined on pages 228 and 229 of Ref. 1, and is known as Holzer's method. The wing data (See Table I.) furnished for the airplane in question assumed the mass moments of inertia I_n of the wing to be concentrated at 7 spanwise stations, the first and last stations being located at the tip and the root respectively. The notation used is the same as that for the fuselage and is demonstrated in Fig. 4.

A positive (climbing) pitching angle at the tip ($n=1$) of one radian was assumed, ($\beta_1 = 1$) and for a given frequency, the inertia torque $\sum_{n=1}^{n=1} I_n \omega^2 \beta_n$ was calculated for station $n=1$. This inertia torque multiplied by the torsional flexibility for panel length l_1 gave the amount of twist or the reduction in angle β between stations $n=1$ and $n=2$. This angle of twist was then subtracted from β_1 to give β_2 , the angular deflection at station $n=2$. Knowing β_2 , the inertia torque at station $n=2$, $\sum_{n=1}^{n=2} I_n \omega^2 \beta_n$ was then calculated.

The remainder of the table was completed in like manner until the residual inertia torque $\sum_{n=1}^{n=7} I_n \omega^2 \beta_n$ (at the wing root) was found. This value of inertia torque was tabulated in Table IV and plotted against ω in Fig. 1. The residual inertia torque is the shaking moment which would be required at the wing root to cause the wing to vibrate torsionally at the assumed frequency ω . In the case

of a rigid fuselage and assuming symmetric torsional vibration, $\sum_{n=1}^{n=7} I_n \omega^2 \beta_n$ must equal zero at a natural torsional frequency of the wing. Consequently these natural frequencies can be found by plotting $\sum_{n=1}^{n=7} I_n \omega^2 \beta_n$ against ω to determine points of intersection with the ω axis as was done in Fig. 1. This calculation was carried out for the first two modes, the final results appearing in Table IX under "Rigid Fuselage".

A sample calculation for $\sum_{n=1}^{n=7} I_n \omega^2 \beta_n$ appears as Table III.

II. CALCULATION FOR THE FLEXIBLE FUSELAGE

(a) Outline of Method Used:

The method used here for the fuselage is that derived in Ref. 2 for the antisymmetric bending of airplane wings. A brief resume of the method follows herewith.

Using the notation demonstrated in Fig. 4, the shear at any station n is given by

$$S_n = \sum_{i=1}^{i=n} m_i \omega^2 y_i \qquad (1)$$

and the bending moment at any station n is given by

$$M_n = \sum_{i=1}^{i=n-1} m_i \omega^2 y_i (x_i - x_n) \qquad (2)$$

The slope at any station $n+1$ is given by

$$\alpha_{n+1} = \alpha_n - S_n v_{F_n} - M_n v_{M_n} \qquad (3)$$

and the deflection at any station $n+1$ is given by

$$y_{n+1} = y_n - \ell_n \alpha_{n+1} - S_n d_{F_n} - M_n d_{M_n} \tag{4}$$

where

v_{F_n} = change in slope from $n+1$ to n due to a unit force at n.

v_{M_n} = change in slope from $n+1$ to n due to a unit moment at n.

d_{F_n} = change in deflection from $n+1$ to n due to a unit force at n.

d_{M_n} = change in deflection from $n+1$ to n due to a unit moment at n.

The method of calculation of these parameters is outlined in section II (d) of this appendix.

Substituting the expressions for S_n and M_n from equations (1) and (2) into equations (3) and (4)

$$\alpha_{n+1} = \alpha_n - \omega^2 v_{F_n} \sum_{i=1}^{i=n} m_i y_i - \omega^2 v_{M_n} \sum_{i=1}^{i=n-1} m_i y_i (x_i - x_n) \tag{5}$$

$$y_{n+1} = y_n - \ell_n \alpha_{n+1} - \omega^2 d_{F_n} \sum_{i=1}^{i=n} m_i y_i - \omega^2 d_{M_n} \sum_{i=1}^{i=n-1} m_i y_i (x_i - x_n) \tag{6}$$

At the end of the fuselage, assume

$$\alpha_1 = \phi$$

$$y_1 = 1$$

so that $\alpha_2 = \phi - v_{F_1} \omega^2 m_1 = \phi - f_2$ (5a)

$$y_2 = 1 - \ell_1 \alpha_2 - \omega^2 d_{F_1} m_1 = 1 + \ell_1 f_2 - d_{F_1} \omega^2 m_1 - \ell_1 \phi = g_2 - \ell_1 \phi \tag{6a}$$

Continuing with equations (5) and (6) in like manner yields

$$\alpha_n = h_n \phi - f_n \tag{7}$$

$$y_n = g_n - k_n \phi \tag{8}$$

where coefficients h_n , f_n , g_n , and k_n are independent of ϕ , one complete set being obtained for each frequency. The method of determination of these coefficients is outlined in section II (b) of this appendix. For the present, these coefficients are assumed to be known for any particular frequency ω .

For antisymmetric bending, the case where the fuselage is being shaken by a shaking moment $M \cos \omega t$ about the elastic axis of the wing, the deflection at the elastic axis is zero $y_b = 0$ and from equation (8) $\phi = \frac{g_b}{k_b}$. With this value of ϕ , all of the deflections y_n may be found by means of equation (8), as can the bending moment at any station n , $\sum_{i=1}^{i=n-1} m_i \omega^2 y_i (x_i - x_n)$.

In the particular calculation with which this paper is concerned, the two quantities desired are the bending moment coming in from the tail or nose M_b , and the slope of the fuselage at the elastic axis α_b .

$$M_b = \sum_{i=1}^{i=b-1} \omega^2 m_i y_i (x_i - x_b) \tag{9}$$

$$= \sum_{i=1}^{i=b-1} m_i \omega^2 g_i (x_i - x_b) - \sum_{i=1}^{i=b-1} m_i \omega^2 k_i \phi (x_i - x_b) \tag{10}$$

Putting $$m_i \omega^2 g_i = G_i \tag{11}$$

and $$m_i \omega^2 k_i = K_i \tag{12}$$

also $$(x_i - x_b) = \sum_{s=i}^{s=b-1} l_s \tag{13}$$

Then $$M_b = \sum_{i=1}^{i=b-1}\left[G_i \sum_{s=i}^{s=b-1} l_s\right] - \sum_{i=1}^{i=b-1}\left[K_i \phi \sum_{s=i}^{s=b-1} l_s\right] \tag{14}$$

But $$\sum_{i=1}^{i=b-1}\left[G_i \sum_{s=i}^{s=b-1} l_s\right] = G_1(l_1 + l_2 + \cdots l_{b-1}) + G_2(l_2 + l_3 + \cdots l_{b-1}) + \cdots G_{b-1}\, l_{b-1}$$

$$= l_1 G_1 + l_2(G_1 + G_2) + l_3(G_1 + G_2 + G_3) + \cdots l_{b-1}(G_1 + G_2 + \cdots G_{b-1})$$

$$= \sum_{i=1}^{i=b-1}\left[l_i \sum_{s=1}^{s=i} G_s\right] \tag{15}$$

and similarly $$\sum_{i=1}^{i=b-1}\left[K_i \phi \sum_{s=i}^{s=b-1} l_s\right] = \phi \sum_{i=1}^{i=b-1}\left[l_i \sum_{s=1}^{s=i} K_s\right]$$

so $$M_b = \sum_{i=1}^{i=b-1}\left[l_i \sum_{s=1}^{s=i} G_s\right] - \phi \sum_{i=1}^{i=b-1}\left[l_i \sum_{s=1}^{s=i} K_s\right] \tag{16}$$

Referring to Table VI (b)

$\sum_{s=1}^{s=i} K_s$ is given by column (2) and

$\sum_{s=1}^{s=i} G_s$ is given by column (6), the second summations occurring in columns (3) and (7) which columns give

$$K_b' = \sum_{i=1}^{i=b-1}\left[l_i \sum_{s=1}^{s=i} K_s\right] \tag{17}$$

and $$G_b' = \sum_{i=1}^{i=b-1}\left[l_i \sum_{s=1}^{s=i} G_s\right] \tag{18}$$

From this it is seen that

$$M_b = G_b' - K_b' \phi \tag{19}$$

and from equation (7) $$\alpha_b = h_b \phi - f_b \tag{20}$$

where h_b and f_b are given on line $n = 8$ under columns (4) and (8) respectively.

The method was repeated for the nose using the same frequency

(Table VI(a)), to obtain M_b' and α_b', and the four values thus determined were tabulated at the top of Table VII.

With an assumed positive (downward) deflection of one inch at the tail the total moment introduced at the elastic axis by the fuselage is given by

$$M_c = (\alpha_b/\alpha_b')\,M_b' + M_b$$

and the slope of the fuselage at the elastic axis is α_b.

(b) Determination of Coefficients

With the original assumptions at the end of the fuselage $\alpha_1 = \phi$ and $y_1 = 1$, from equations (7) and (8)

$$\alpha_n = h_n\phi - f_n$$

$$y_n = g_n - k_n\phi$$

it is obvious that

$h_1 = 1$ $\quad f_1 = 0$ $\quad g_1 = 1$ $\quad k_1 = 0$, and

from equations (5a) and (6a)

$$\alpha_2 = \phi - f_2$$

$$y_2 = g_2 - \ell_1\phi$$

are obtained $h_2 = 1$ and $k_2 = \ell_1$.

Substituting equations (7) and (8) into equations (5) and (6),

$$h_{n+1}\phi - f_{n+1} = h_n\phi - f_n - \omega^2 v_{F_n}\sum_{i=1}^{i=n} m_i(g_i - k_i\phi) - \omega^2 v_{M_n}\sum_{i=1}^{i=n-1} m_i(g_i - k_i\phi)(x_i - x_n) \qquad (21)$$

$$g_{n+1} - k_{n+1}\phi = g_n - k_n\phi - \ell_n h_{n+1}\phi + \ell_n f_{n+1} - \omega^2 d_{F_n}\sum_{i=1}^{i=n} m_i(g_i - k_i\phi) - \omega^2 d_{M_n}\sum_{i=1}^{i=n-1} m_i(g_i - k_i\phi)(x_i - x_n) \qquad (22)$$

By equating terms containing ϕ and those not containing ϕ on the two sides of each of equations (21) and (22), the following equations are obtained:

Equating coefficients of ϕ :

$$h_{n+1} = h_n + \omega^2 v_{F_n} \sum_{i=1}^{i=n} m_i k_i + \omega^2 v_{M_n} \sum_{i=1}^{i=n-1} m_i k_i (x_i - x_n) \qquad (23)$$

$$k_{n+1} = k_n + l_n h_{n+1} - \omega^2 d_{F_n} \sum_{i=1}^{i=n} m_i k_i - \omega^2 d_{M_n} \sum_{i=1}^{i=n-1} m_i k_i (x_i - x_n) \qquad (24)$$

Equating constant terms:

$$f_{n+1} = f_n + \omega^2 v_{F_n} \sum_{i=1}^{i=n} m_i g_i + \omega^2 v_{M_n} \sum_{i=1}^{i=n-1} m_i g_i (x_i - x_n) \qquad (25)$$

$$g_{n+1} = g_n + l_n f_{n+1} - \omega^2 d_{F_n} \sum_{i=1}^{i=n} m_i g_i - \omega^2 d_{M_n} \sum_{i=1}^{i=n-1} m_i g_i (x_i - x_n) \qquad (26)$$

Using substitutions and relations developed in (11), (12), (13), (15) and (16):

$$h_{n+1} = h_n + v_{F_n} \sum_{i=1}^{i=n} K_i + v_{M_n} \sum_{i=1}^{i=n-1} \left[l_i \sum_{s=1}^{s=i} K_s \right] \qquad (27)$$

$$k_{n+1} = k_n + l_n h_{n+1} - d_{F_n} \sum_{i=1}^{i=n} K_i - d_{M_n} \sum_{i=1}^{i=n-1} \left[l_i \sum_{s=1}^{s=i} K_s \right] \qquad (28)$$

$$f_{n+1} = f_n + v_{F_n} \sum_{i=1}^{i=n} G_i + v_{M_n} \sum_{i=1}^{i=n-1} \left[l_i \sum_{s=1}^{s=i} G_s \right] \qquad (29)$$

$$g_{n+1} = g_n + l_n f_{n+1} - d_{F_n} \sum_{i=1}^{i=n} G_i - d_{M_n} \sum_{i=1}^{i=n-1} \left[l_i \sum_{s=1}^{s=i} G_s \right] \qquad (30)$$

All the coefficients h_n , k_n , f_n , and g_n can be found by progressive calculation with the aid of a table (such as Tables VI(a) and VI(b)) based on these equations.

(c) Procedure Used in Filling out Tables VI(a) and VI(b)

Parameters v_{F_n} , v_{M_n} , d_{F_n} , and d_{M_n} were calculated (see section II(d) of this appendix) and written in the spaces indicated. Likewise the values for ℓ_n were entered in the tables.

Then, for a given frequency, values of $m_n\omega^2$ were calculated and entered in the appropriate column.

Next, in line $n=1$, the following values were entered in columns (1), (4) and (5) respectively:

$k_1 = 0$ $\qquad\qquad\qquad\qquad$ $h_2 = 1$ $\qquad\qquad\qquad\qquad$ $g_1 = 1$

and in line $n=2$ under column (1): $\qquad k_2 = \ell_1$

The table was then worked across from left to right starting with line $n=1$ then proceeding with line $n=2$ etc., each step being indicated in the column heading.

In computing values to enter in columns (1), (3), (5) and (7), one must remember to use information appearing in the preceeding line.

The remainder of the steps are self explanatory, the desired quantities of the calculation being M_b and α_b .

(d) Calculation of Parameters v_{F_n} , v_{M_n} , d_{F_n} , and d_{M_n}.

The fuselage data received (Table II.) indicated bending moments of inertia equal to zero at each end of the fuselage, but in order to more nearly approximate the probable moment of inertia distribution in the regions from the extreme ends to the next stations inboard, a trapezoidal distribution over these regions was assumed in both cases with the end ordinates approximately half the value of the next ordinates inboard. The assumed values were

I_1 (TAIL) $= 500$ in.^{4} $\qquad\qquad\qquad$ I_1 (NOSE) $= 3,500$ in.^{4}

A fuselage stiffness curve (such as Fig. 5) could be drawn for the given and assumed (end) values of bending moment of inertia, where the bending moment of inertia is plotted against fuselage distance as abscissa, such that the areas between succeeding stations would be trapezoids. If I_0 is taken as a convenient reference value of the bending moment of inertia for the fuselage as a whole, then at any point between stations n and $n+1$

$$I = I_0\left(a_n + \frac{b_n}{\ell_n}\xi\right)$$

Using the moment area method:

$$V_{M_n} = \int_0^{\ell_n} \frac{d\xi}{EI_0\left(a_n + \frac{b_n}{\ell_n}\xi\right)} = \frac{1}{EI_0}\frac{\ell_n}{b_n}\log_e\left[a_n + \frac{b_n}{\ell_n}\xi\right]_0^{\ell_n} = \frac{1}{EI_0}\frac{\ell_n}{b_n}\log_e\left[\frac{a_n+b_n}{a_n}\right] \qquad (31)$$

$$V_{F_n} = d_{M_n} = \int_0^{\ell_n} \frac{\xi\, d\xi}{EI_0\left(a_n + \frac{b_n}{\ell_n}\xi\right)} = \frac{1}{EI_0}\frac{\ell_n^2}{b_n^2}\left[b_n - a_n\log_e\left(\frac{a_n+b_n}{a_n}\right)\right] \qquad (32)$$

$$d_{F_n} = \int_0^{\ell_n} \frac{\xi^2\, d\xi}{EI_0\left(a_n + \frac{b_n}{\ell_n}\xi\right)} = \frac{1}{EI_0}\frac{\ell_n^3}{b_n^3}\left[\frac{1}{2}b_n^2 + a_n^2\log_e\left(\frac{a_n+b_n}{a_n}\right) - a_n b_n\right] \qquad (33)$$

Referring to Tables V(a) and (b), EI_0 was taken as 10^{10}. Columns (1) and (2) were filled in with values of ℓ_n and a_n from the data furnished. The value $(a_n + b_n)$ in column (3) is determined from

$$a_{n+1} = a_n + b_n$$

Due to lack of availability of a seven place table of natural logarithms, common logarithms were used and values converted to natural logarithms in columns (8) and (9) by the relation

$$\log_e\left(\frac{a_n+b_n}{a_n}\right) = 2.302585\,\log_{10}\left(\frac{a_n+b_n}{a_n}\right)$$

The remainder of the table is self explanatory and follows from

equations (31), (32) and (33). The desired parameters appear in columns (8), (12) and (18).

(e) Procedure Used in Filling out Table VII.

Again using the same frequency as was used in Tables VI(a) and VI(b), Holzer's calculation (as explained in section I of this appendix) was repeated in Table VII, the desired quantities being the angular deflection of the wing at the root β_7 and the inertia torque at the next station outboard from the root $\sum_{n=1}^{n=6} I_n \omega^2 \beta_n$. This inertia torque adjusted so as to make the angular deflection of the wing at the root equal to the slope of the fuselage at the elastic axis is $\left\{\sum_{n=1}^{n=6} I_n \omega^2 \beta_n\right\}(\alpha_b/\beta_7)$, and this adjusted inertia torque added to half the bending moment introduced at the elastic axis by the fuselage $\frac{1}{2} M_c$ gives the residual torque acting at the wing root which would be required to make the wing vibrate torsionally at the chosen frequency.

$$M = \frac{1}{2} M_c + \left\{\sum_{n=1}^{n=6} I_n \omega^2 \beta_n\right\}(\alpha_b/\beta_7)$$

At a natural frequency of the system this residual torque is zero. Table VIII gives a tabulation of the results of this calculation. Fig. 1 shows a plot of this residual torque against ω and the natural frequencies (first and second modes) occur when this curve crosses the ω axis. The final results are tabulated in Table IX under "Flexible Fuselage".

REFERENCES

Reference 1. Den Hartog; "Mechanical Vibrations", McGraw-Hill Book Company, Inc.; New York and London, 1940.

Reference 2. N. O. Mykelstad; "A New Method of Calculating Natural Modes of Uncoupled Bending Vibration of Airplane Wings and Other Types of Beams". Journal of the Aeronautical Sciences, Vol. 11, No. 2, April, 1944.

NO OF STATION n	DISTANCE OF STATION FROM WING ROOT (inches)	PANEL LENGTHS FROM n TO $n+1$ ℓ_n (inches)	MASS MOMENT OF INERTIA I_n (lb. in sec.^{2})	TORSIONAL FLEXIBILITY PER INCH OF ℓ_n (rad/in.lb./in)	TORSIONAL FLEXIBILITY OF ℓ_n - $1/K_n'$ (rad/in lb)
1	638	90	34	1.41×10^{-9}	126.9×10^{-9}
2	548	120	287	0.487×10^{-9}	58.44×10^{-9}
3	428	121	536	$0\ 103 \times 10^{-9}$	$12\ 463 \times 10^{9}$
4	307	90	62,000	0.093×10^{-9}	8.37×10^{-9}
5	217	84	1,300	$0\ 026 \times 10^{-9}$	2.184×10^{-9}
6	133	133	85,000	0.026×10^{-9}	3.458×10^{-9}
7	0	—	1,000,000	—	—

TABLE I - WING DATA

NO OF STATION n NOSE	NO OF STATION n TAIL	DISTANCE FROM FUSELAGE NOSE (inches)	PANEL LENGTHS FROM n TO $n+1$ ℓ_n (inches)	FUSELAGE BENDING MOMENT OF INERTIA I_n (inches^4)	FUSELAGE WEIGHT W_n (pounds)	FUSELAGE MASS $W_n/386$ m_n (lb. sec^2/in)
1		0	74.50	0	513.1	1.329 274
2		74.50	76.50	6,911	1088.9	2.820 984
3		151.00	74.00	5,429	2135.0	5.531 088
4		225.00	33.56	8,572	1810.9	4.691 450
5		258.56	50.44	11,415	660.4	1.710 880
6	9	* 309.00		13,800	629.7	1.631 347
	8	349.50	40.50	16,364	935.0	2.422 279
	7	438.00	88.50	13,773	779.9	2.020 466
	6	506.00	68.00	17,710	1,015.0	2.629 533
	5	594.00	88.00	11,037	292.9	0.758 808
	4	683.00	89.00	3,610	209.0	0.541 450
	3	714.00	31.00	2,772	872.3	2.259 844
	2	758.00	44 00	1,000	778.3	2.016 321
	1	791.50	33.50	0	412.9	1.069 689

* Taken as elastic axis

TABLE [illegible] - FUSELAGE DATA

$\omega = 33.67215$				$\omega^2 = 1133.8136$		
n	I	$I\omega^2 10^{-6}$	β	$\Sigma I \omega^2 \beta 10^{-6}$	$1/K'_n 10^4$	$1/K'_n \Sigma I \omega^2 \beta 10^3$
1	34	0.03854966	1.0000000	0.0385497	126.9	4.891952
2	287	0.3254045	0.9951080	0.3623623	58.44	21.17645
3	536	0.6077241	0.9739316	0.9542440	12.463	11.89274
4	62,000	70.29644	0.9620389	68.58215	8.37	574.0326
5	1,300	1.473958	0.3880063	69.15405	2.184	151.0325
6	85,000	96.37416	0.2369738	91.99220	3.458	318.1090
7	1,000,000	1133.8136	0.0811352	+ 0.000007		

TABLE III - HOLZER'S CALCULATION FOR RIGID FUSELAGE

ω	ω^2	$\sum_{n=1}^{n=7} I_n \omega^2 \beta_n \times 10^{-6}$
20.0000	400	270.0200
22.3607	500	284.8015
28.2843	800	222.0618
31.6228	1000	105.3761
33.3916	1115	15.9933
33.6719	1133.8	0.0117
* 33.67215	1133.814	+0.000007
33.6898	1135	−1.02085
37.4170	1400	261.2742
45.8260	2100	1126.8828
54.7720	3000	2116.6199
63.2460	4000	2223.6800
71.4140	5100	136.1946
71.6909	5139.58	− 5.9569566
* 71.70349	5141.39	+ 0.1016590
72.5000	5250	379.9307
74.1620	5500	1379.9740
77.4600	6000	3937.3600

* NATURAL FREQUENCIES

TABLE IV - RESULTS OF RIGID FUSELAGE CALCULATION

n	DISTANCE FROM NOSE	1 l	2 a	3 $a + b =$ $(2)_{n+1}$	4 $(a+b)/a =$ $(3)/(2)$	5 $\log_{10}(a+b)/a =$ $\log_{10}(4)$	6 $b =$ $(3)-(2)$	7 $l/b =$ $(1)/(6)$	8 $V_m \times 10^{10} =$ $l/b \log_e(a+b)/a$ $= 2.302585 \times (5) \times (7)$	9 $a \log_e(a+b)/a =$ $2.302585 \times (5) \times (2)$
1	0	74.5	3.500	6.911	1.9745714	0.2954728	3.411	21.84110	14.85962	2.381229
2	74.5	76.5	6.911	5.429	0.7855593	9.8951790-10	-1.482	-51.61943	12.45883	-1.668034
3	151.0	74.0	5.429	8.572	1.5789280	0.1983623	3.143	23.54438	10.75381	2.479675
4	225.0	33.56	8.572	11.415	1.3316612	0.1243937	2.843	11.80443	3.381109	2.455253
5	258.56	50.44	11.415	13.800	1.2089356	0.08[illegible]4032	2.385	21.14885	4.012791	2.165887
6	309.0		13.800							

n	10 $b - a \log_e(a+b)/a$ $= (6) - (9)$	11 $(l/b)^2 =$ $(7) \times (7)$	12 $V_r \times 10^{10} = d_m \times 10^{10} =$ $(l/b)^2 [b - a \log_e (a+b)/a]$ $= (10) \times (11)$	13 $b^2/2 =$ $(6) \times (6)/2$	14 $a^2 \log_e(a+b)/a$ $= (2) \times (9)$	15 $ab =$ $(2) \times (6)$	16 $b^2/2 + a^2 \log_e \frac{a+b}{a} - ab$ $= (13) + (14) - (15)$	17 $(l/b)^3 =$ $(7) \times (11)$	18 $d_\pi \times 10^{10} =$ $(l/b)^3 [\frac{b^2}{2} + a^2 \log_e \frac{a+b}{a} - ab]$ $= (16) \times (17)$
1	1.029771	477.0337	491.2355	5.817461	8.334302	11.93850	2.213763	10,418.94	23,059.85
2	0.186034	2,664.566	495.6999	1.098162	-11.52778	-10.24210	-0.187518	-137,543.4	25,791.86
3	0.663325	554.3380	367.7063	4.939225	13.46216	17.06335	1.338035	13,051.55	17,463.43
4	0.387747	139.3446	54.03045	4.041325	21.04643	24.37020	0.717555	1,644.884	1,180.295
5	0.219113	447.2737	98.00348	2.844113	24.72360	27.22478	0.342933	9,459.323	3,243.914

TABLE V(a) - CALCULATION OF PARAMETERS-NOSE

n	DISTANCE FROM NOSE	1	2	3	4	5	6	7	8	9
		l	a	$a+b = (2)_{n+1}$	$(a+b)/a = (3)/(2)$	$\log_{10}(a+b)/a = \log_{10}(4)$	$b = (3)-(2)$	$l/b = (1)/(6)$	$V_m \times 10^{10} =$ [illegible] $\log_e(a+b)/a = 2.302585 \times (5) \times (7)$	$a \log_e(a+b)/a = 2.302585 \times (5) \times (2)$
1	741.5	33.5	0.500	1.000	2.000000	0.3010300	0.500	67.00000	46.44086	0.3465736
2	758.0	44.0	1.000	2.772	2.772000	0.4427932	1.772	24.83070	25.31661	1.019569
3	714.0	31.0	2.772	3.610	1.302309	0.1147139	0.838	36.99284	9.7712[illegible]	0.7321919
4	683.0	890	3.610	11.037	3.057341	0.4853439	7.427	11.98330	13.39189	4.034340
5	594.0	880	11.037	17.710	1.604605	0.2053675	6.673	13.18747	6.236040	5.219134
6	506.0	680	17.710	13.773	0.7776962	9.8908100-10	-3.937	-17.27203	4.342523	-4.452636
7	438.0	885	13.773	16.364	1.188122	0.0748609	2.591	34.15670	5.8877[illegible]	2.374102
8	349.5	40.5	16.364	13.800	0.8433146	9.9259896-10	-2.564	-15.79563	2.[illegible]1816	-2.788674
9	309.0		13.800							

n	10	11	12	13	14	15	16	17	18
	$b - a\log_e(a+b)/a = (6)-(9)$	$(l/b)^2 = (7)\times(7)$	$V_F \times 10^{10} = d_m \times 10^{10} = (l/b)^2[b - a\log_e(a+b)/a] = (10)\times(11)$	$b^2/2 = (6)\times(6)/2$	$a^2\log_e(a+b)/a = (2)\times(9)$	$ab = (2)\times(6)$	$b^2/2 + a^2\log_e\frac{a+b}{a} - ab = (13)+(14)-(15)$	$(l/b)^3 = (7)\times(11)$	$d_F \times 10^{10} = (l/b)^3[b^2/2 + a^2\log_e\frac{a+b}{a} - ab] = (16)\times(17)$
1	0.1534064	4,489.000	688.7311	0.125000	0.1732868	0.250000	0.0482868	300,763.0	14,522.88
2	0.7524310	616.5636	463.9216	1.569992	1.019569	1.772000	0.8175610	15,309.71	12,516.62
3	0.1058081	1,368.470	144.7952	0.351122	2.029636	2.322936	0.0578220	50,623.60	2,9[illegible]
4	3.392[illegible]	143.5996	487.1847	27.58016	14.56397	26.81147	15.33266	1,720.798	26,384.41
5	1.453867	173.9094	252.8411	22.26446	57.60358	73.64990	6.218140	2,293.425	14,260.84
6	0.5156358	298.3232	153.8261	7.749984	-12.85618	-69.72427	-1.381926	-5,152.649	7,120.580
7	0.2168984	1,166.680	253.0510	3.356640	3[illegible]64850	35.68584	0.369300	39,849.93	14,716.58
8	0.224[illegible]743	249.5020	56.05669	3.287048	-45.63387	-41.95730	-0.3895220	-3,941.042	1,535.123

TABLE V(b) - CALCULATION OF PARAMETERS-TAIL

$\omega = \underline{31.42148}$ $\omega^2 = \underline{987.3095}$ $f = \frac{60}{2\pi}\omega = 300.3$

NOSE

n	$\frac{m\omega^2}{10^7}$	(1) $k = l(4) + (1) - d_F(2) - d_M(3)$ (for $n-1$)	$d_F\,10^8$	(2) $K10^{-7} = \Sigma \frac{m\omega^2}{10^7}(1)$	$d_M 10^8 = V_F 10^8$	(3) $K'10^{-7} = \Sigma l(2)$ (for $n-1$)	$V_M\,10^8$	(4) $k_{n+1} = k_n + V_F(2) + V_M(3)$	l	$y' = g - k\phi$
1	.0001312405	0	230.5985	0	4.912355	0	.1485962	1	74.5	1
2	.0002785184	74.5	257.9186	.02074962	4.956999	0	.1245883	1.010286	76.5	.719119
3	.0005460896	151.2517	174.6343	.1033466	3.677063	1587346	.1075381	1.065357	74.0	.442298
4	.0004631913	2276997	11.80295	.2088151	.5403045	9.234994	.03381109	1.107864	33.56	.208112
5	.0001689168	264.1342	32.43914	.2534318	.9800348	16.24.283	.04012791	1.197880	50.44	.117336
6		322.1413				29 02593				0
7										
8										
9										

$$\phi = \frac{g_b}{k_b} = \frac{1.222225}{322.1413} = \underline{.003794064}$$

$$M_b' = G_b' - K_b'\phi = \underline{+1.347803 \times 10^6}$$

$$\alpha_b' = k_b\phi - f_b = \underline{+.002087593}$$

n	$\frac{m\omega^2}{10^7}$	(5) $g = l(8) + (5) - d_F(6) - d_M(7)$ (for $n-1$)	$d_F\,10^8$	(6) $G10^{-7} = \Sigma \frac{m\omega^2}{10^7}(5)$	$d_M 10^8 = V_F 10^8$	(7) $G'10^{-7} = \Sigma l(6)$ (for $n-1$)	$V_M\,10^8$	(8) $f_{n+1} = f_n + V_F(6) + V_M(7)$	l	$k\phi$
1	0001312405	1	230.5985	0001312405	4.912355	0	.1485962	.00006446999	74.5	0
2	.0002785184	1.001777	257.9186	.0004102537	4.956999	.009777417	.1245883	.0003896479	76.5	.282658
3	.0005460896	1.016157	174.6343	.0009651666	3.677063	04116183	.1075381	.001187192	74.0	.573859
4	0004631913	1.072019	11.80295	.001461716	.5403045	.1125842	.03381109	.001646829	33.56	.863907
5	.0001689168	1.119478	32.43914	.001650815	.9800348	.1616394	.04012791	.002457240	50.44	1.002142
6		1.222225				.2449065		.004544833		1.222225
7										
8										
9										

TABLE VI([illegible]) - CALCULATION OF MOMENT, DEFLECTION AND SLOPE - NOSE

$\omega = 314.2148$ $\omega^2 = 98730.95$ $f = \frac{60}{2\pi}\omega = 3000.3$ TAIL

n	$\frac{m\omega^2}{10^7}$	(1) $k = \ell(4) + (1) - d_F(2) - d_M(3)$ (for n-1)	$d_F 10^8$	(2) $K 10^{-7} = \sum \frac{m\omega^2}{10^7}(1)$	$d_M 10^8 = V_F 10^8$	(3) $K' 10^{-7} = \sum \ell(2)$ (for n-1)	$V_M 10^8$	(4) $k_{n+1} = k_n + V_F(2) + V_M(3)$	ℓ	$y = g - k\phi$
1	.0001056114	0	145.2288	0	6.887311	0	.4644086	1	33.5	1
2	.0001990733	33.5	125.1662	.006668956	4.639216	0	.2531661	1.003094	44.0	-0.879176
3	.0002231165	77.55266	29.27158	.02397223	1.447952	.2934340	.09771233	1.009432	31.0	0.727011
4	.00005345787	108.7324	263.8441	.02978483	4.871847	1.036573	.1339189	1.037824	89.0	0.626639
5	.00007491783	199.8079	142.6084	.04475401	2.528411	3.687423	.06236040	1.072135	88.0	0.377753
6	.0002596163	292.5852	71.20580	.1207139	1.538261	7.625776	.04342523	1.123819	68.0	0.183697
7	.0001994825	366.9723	147.1658	.1939185	2.530510	15.83432	.05887713	1.266118	88.5	+0.071688
8	.0002391539	472.1631	15.35123	.3068381	.5605669	32.99610	.02691816	1.372138	40.5	-0.000047
9		525.4140				45.42304				0

$$\phi = \frac{g_b}{k_b} = \frac{1.909168}{525.4140} = .003633646$$

$$M_b = G'_b - K'_b\phi = +2.270696 \times 10^6$$

$$\alpha_b = k_b\phi - f_b = -.000305154$$

n	$\frac{m\omega^2}{10^7}$	(5) $g = \ell(8) + (5) - d_F(6) - d_M(7)$ (for n-1)	$d_F 10^8$	(6) $G 10^{-7} = \sum \frac{m\omega^2}{10^7}(5)$	$d_M 10^8 = V_F 10^8$	(7) $G' 10^{-7} = \sum \ell(6)$ (for n-1)	$V_M 10^8$	(8) $f_{n+1} = f_n + V_F(6) + V_M(7)$	ℓ	$k\phi$
1	.0001056114	1	145.2288	.0001056114	6.887311	0	.4644086	.00007273786	33.5	0
2	.0001990733	1.000903	125.1662	.0003048645	4.639216	.003537982	.2531661	.0003037408	44.0	.121727
3	.0002231165	1.008810	29.27158	.0005299467	1.447952	.01695202	.09771233	.0005461167	31.0	.281799
4	.00005345787	1.021734	263.8441	.0005845664	4.871847	.03338037	.1339189	.001277935	89.0	.395095
5	.00007491783	1.103784	142.6084	.0006672595	2.528411	.08540678	.06236040	.001979246	88.0	.726031
6	.0002596163	1.246848	71.20580	.0009409615	1.538261	.1441256	.04342523	.002757550	68.0	1.063151
7	.0001994825	1.405135	147.1658	.001271261	2.530510	.2115110	.05887713	.004324560	88.5	1.333447
8	.0002391539	1.715627	15.35123	.001681560	.5605669	.3240176	.02691816	.005241018	40.5	1.715674
9		1.909168				.3921208		.004985864		1.909168

TABLE VI(b) - CALCULATION OF MOMENT, SLOPE AND DEFLECTION - TAIL

$\omega = \underline{31.42148}$ $\qquad \omega^2 = \underline{987.3095}$

$\alpha_b\,(\text{TAIL}) = \underline{-0.000305154}$

$\alpha_b'\,(\text{NOSE}) = \underline{+0.002087593}$ $\qquad \dfrac{\alpha_b}{\alpha_b'} = \underline{-0.1461750}$

$$M_c = \underset{(-0.197015)}{\frac{\alpha_b}{\alpha_b'} M_b'} + \underset{(+2.270696)}{M_b} = \underline{+2.073681 \times 10^6}$$

$M_b\,(\text{TAIL}) = \underline{+2.270696 \times 10^6}$

$M_b'\,(\text{NOSE}) = \underline{+1.347803 \times 10^6}$

WING STATION n	$I_\circ$	$I\omega^2 10^{-6}$	β	$\Sigma I\omega^2\beta 10^{-6}$	$\frac{1}{k_n'} 10^9$	$\frac{1}{k_n'} \Sigma I\omega^2\beta 10^3$	
1 (638)	34	0.03356852	1	0.03356852	126.9	4.259845	
2 (548)	287	0.2833578	0.9957402	0.3157193	58.44	18.45063	
3 (428)	536	0.5291979	0.9772896	0.8328989	12.463	10.38042	
4 (307)	62,000	61.21319	0.9669092	60.02050	8.37	502.3715	
5 (217)	1,300	1.283502	0.4645377	60.61674	2.184	132.3869	
6 (133)	85,000	83.92131	0.3321508	88.49127	3.458	306.0028	
7 (0)			0.0261480				

$$M = M_c/2 + (\alpha_b/\beta_7) \times \sum_{n=1}^{n=6} I\omega^2\beta = 1.0368405 - 1.032716 = +0.004125$$

TABLE VII - CALCULATION OF RESIDUAL TORQUE

ω	ω^2	$M \times 10^{-6}$
20.00000	400	0.7079923
28.28427	800	1.0198740
29.15476	850	0.9994380
31.14482	970	0.4361850
31.32092	981	0.2045320
* 31.42148	987.3095	+0.0041245
31.43247	988	-0.0223850
31.62278	1000	0.7522720
31.93740	1020	-22.8464700
32.24900	1040	+3.8171720
32.86340	1080	2.1023500
33.67215	1133.8136	1.7240340
38.73000	1500	1.1297870
43.58900	1900	0.0230745
43.64631	1905	+0.0019884
*43.65170	1905.4715	0 BY INTERPOLATION
44.72100	2000	-0.4468940

* NATURAL FREQUENCIES

TABLE VIII - RESULTS OF FLEXIBLE FUSELAGE CALCULATION

MODE	RIGID FUSELAGE ω (RADIANS/SEC)	FLEXIBLE FUSELAGE ω (RADIANS/SEC)	PERCENTAGE DECREASE
FIRST	33.67215	32.42148	6.68
SECOND	71.70349	43.65171	39.1

TABLE IX - FINAL RESULTS

Number of station n		Distance from fuselage nose	First mode $\omega = 31.42148$		Second mode $\omega = 43.6463057$	
Nose	Tail		y'	Deflection for nose $y' \times (-\alpha_b/\alpha_b')$ for tail y	y'	Deflection for nose $y' \times (-\alpha_b/\alpha_b')$ for tail y
1		0	1.00000	0.146175	1.000000	1.811893
2		74.50	0.719119	0.105117	0.684664	1.240538
3		151.00	0.442298	0.064653	0.383022	0.693995
4		225.00	0.208112	0.030421	0.154186	0.279369
5		258.56	0.117336	+ 0.017152	0.078781	+0.142743
6	9	309.00	0	0	0	0
	8	349.50		- 0.000047		- 0.062598
	7	438.00		+ 0.071688		- 0.077684
	6	506.00		0.183697		+ 0.009384
	5	594.00		0.377753		0.209373
	4	683.00		0.626639		0.507356
	3	714.00		0.727011		0.636222
	2	758.00		0.879176		0.837362
	1	791.50		1.000000		1.000000

TABLE X - FUSELAGE DEFLECTIONS

NUMBER OF WING STATION n	DISTANCE OF STATION FROM ROOT	FIRST MODE $\omega = 31.42148$		SECOND MODE $\omega = 43.6463057$	
		β	ANGULAR DEFLECTION (ADJUSTED) $\beta \times (\alpha_b/\beta_T) \times 10^2$	β	ANGULAR DEFLECTION (ADJUSTED) $\beta \times (\alpha_b/\beta_1) \times 10^2$
1	638	1.000000	-1.167	1.000000	+0.413
2	548	0.999740	1.163	0.991781	0.409
3	428	0.977290	1.141	0.956307	0.395
4	307	0.966909	1.128	0.936572	+0.387
5	217	0.464538	0.542	-0.002559	-0.001
6	133	0.332151	0.388	-0.247594	-0.102
7	0	0.026148	-0.031	-0.496930	-0.205

TABLE XI - WING ANGULAR DEFLECTIONS

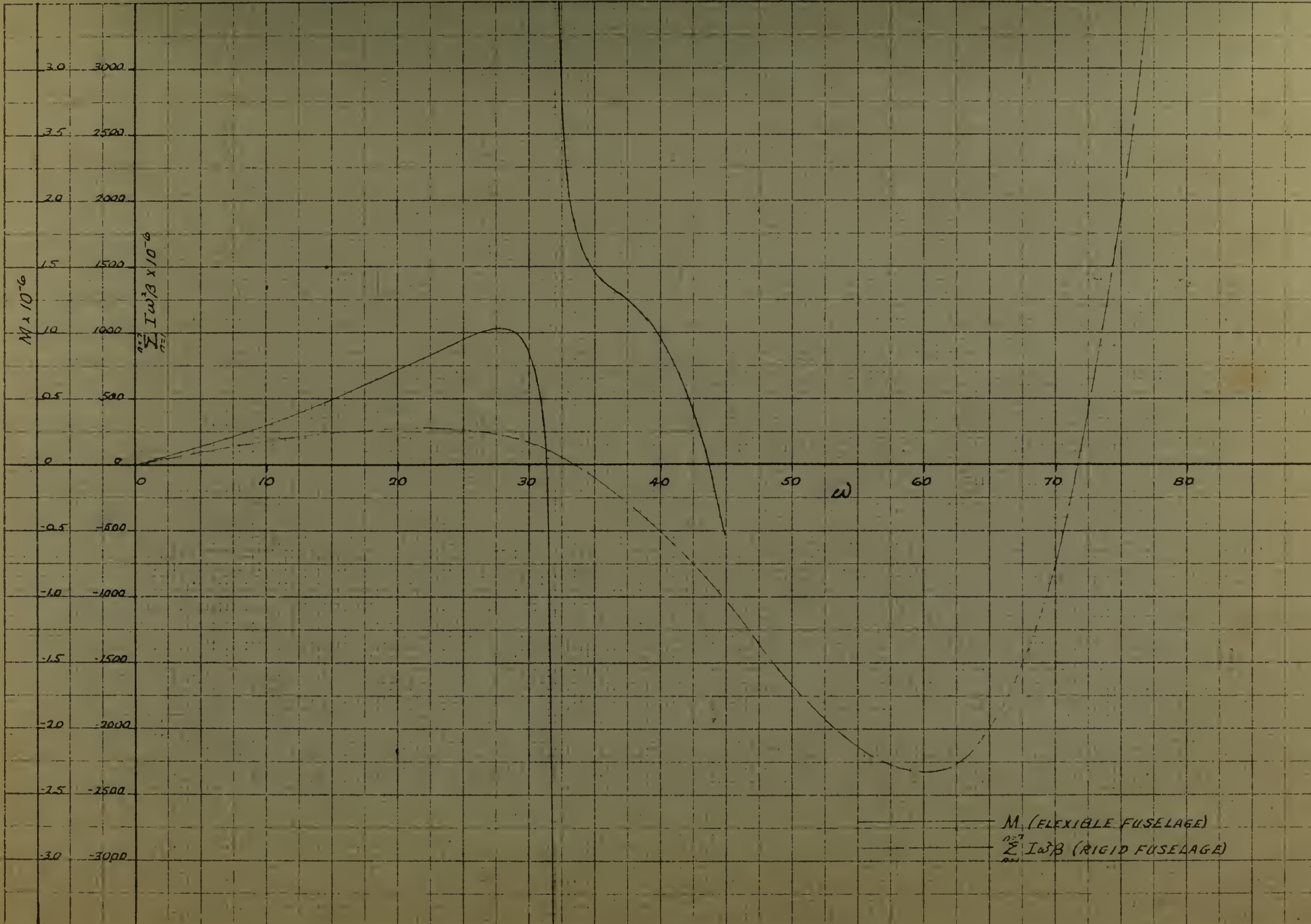


FIG. 1 - RESIDUAL TORQUE VS. FREQUENCY FOR RIGID AND FLEXIBLE FUSELAGE

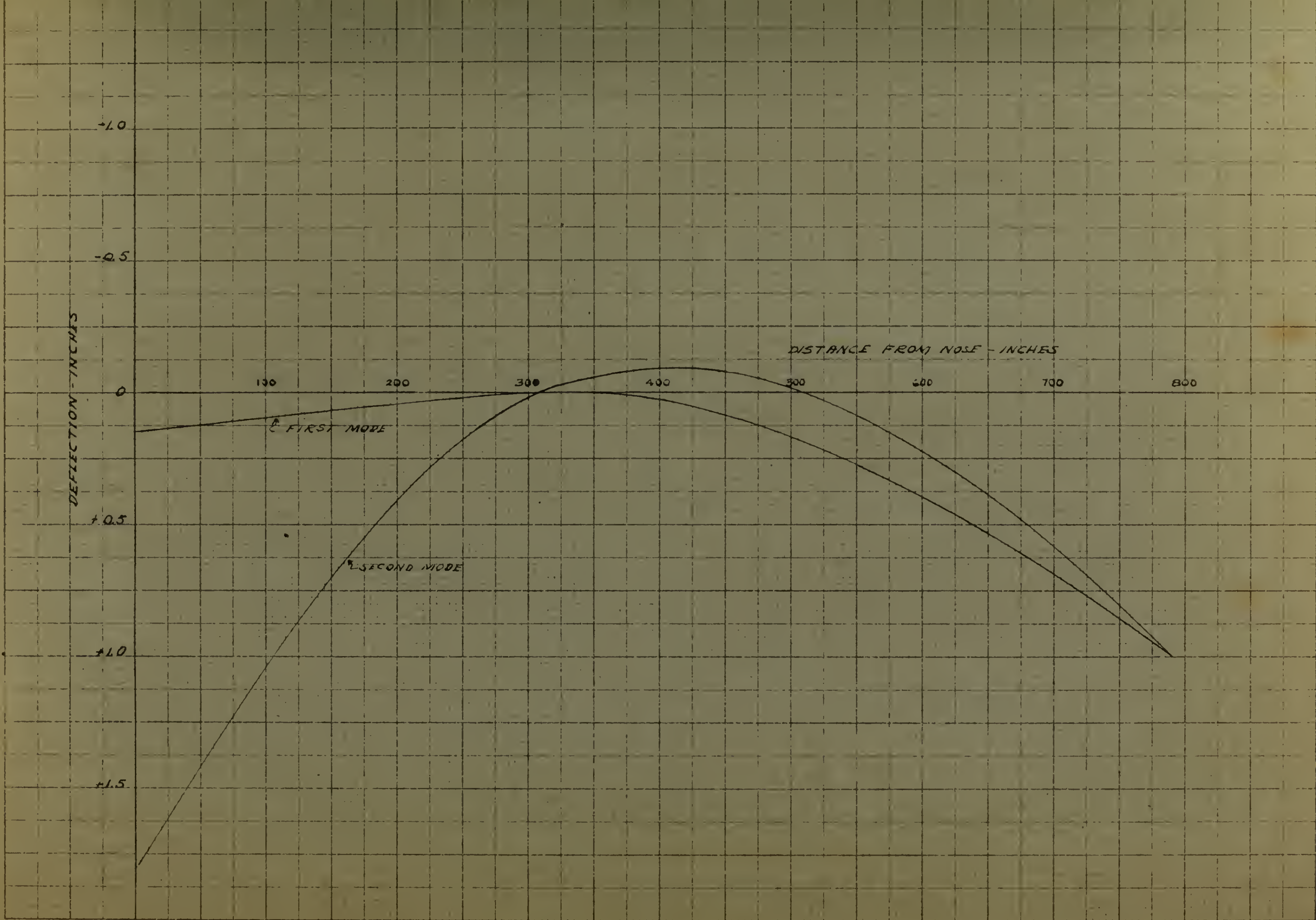


FIG. 2 - FUSELAGE DEFLECTION CURVES

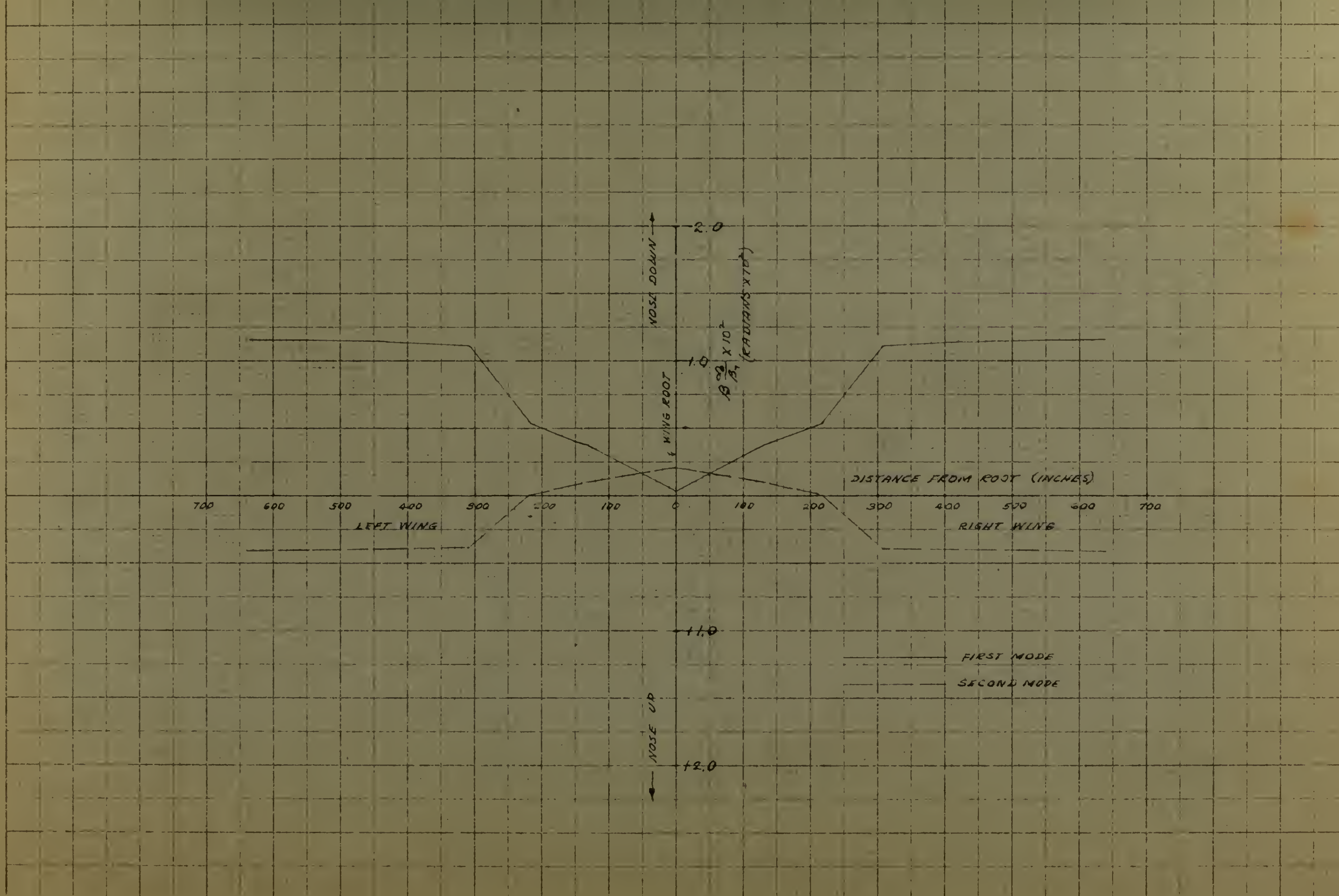


FIG. 3 - WING ANGULAR DEFLECTION CURVES

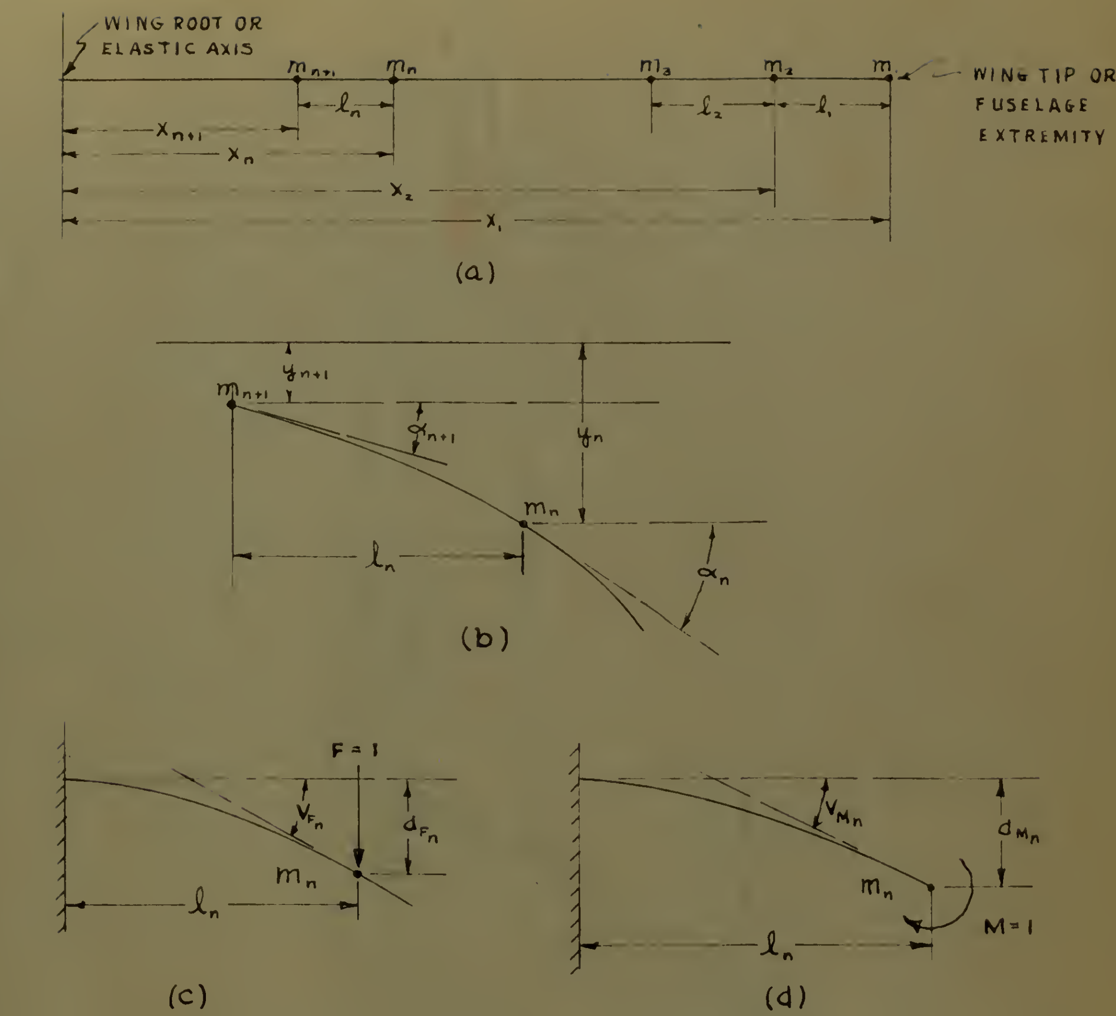


FIG. 4 - SKETCHES SHOWING NOTATION USED IN BENDING CALCULATIONS

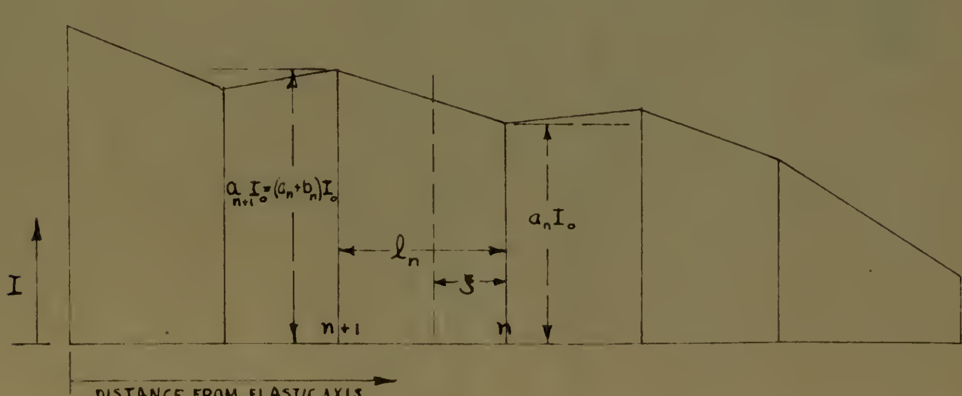
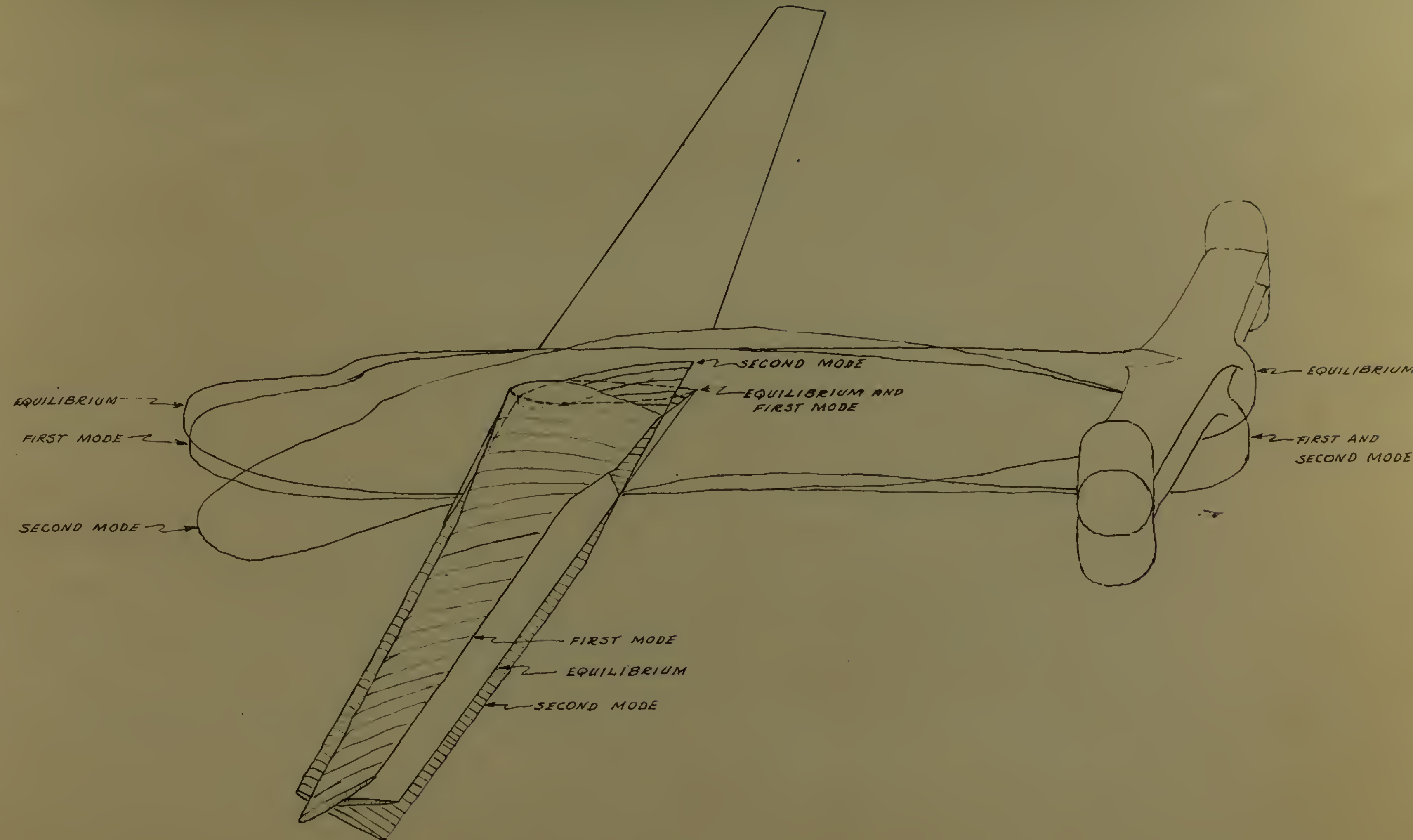


FIG. 5 - SAMPLE FUSELAGE STIFFNESS CURVE AND NOTATION USED



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